Series 4 - MATH 452

March 2025

Problem 4

Consider a Discontinuous Collocation Method (DCM) with collocation polynomial $u(t) \in \mathbb{P}^{s-2}$ such that:

$$\begin{cases} u(t_0) = y_0 - hb_1(u'(t_0) - f(t_0, u(t_0)) \\ u'(t_0 + c_i h) = f(t_0 + c_i h, u(t_0 + c_i h) \quad \text{for} \quad i = 2, ..., s-1 \end{cases} \tag{1}$$

Then the method is given by:

$$y_1 = u(t_0 + h) - hb_s(u'(t_0 + h) - f(t_0 + h, u(t_0 + h)) \tag{2}$$

We first define the following:

$$K_i := f(t_0 + c_i h, u(t_0 + c_i h)) \tag{3}$$

Note that $K_i = u'(t_0 + c_i h)$ for all $i = 1, ..., s-1$. Therefore, we rewrite $u'(t)$ by means of Legendre polynomials of order $s-3$ that is:

$$u'(t_0 + rh) = \sum_{i=2}^{s-1} l_i(r) K_i \tag{4}$$

Where

$$l_i(r) = \prod_{k=2, k\neq i}^{s-1} \frac{r - c_k}{c_i - c_k}$$

By the fundamental theorem of calculus we derive the following:

$$\begin{aligned} u(t_0 + c_i h) &= u(t_0) + \int_{t_0}^{t_0 + c_i h} u'(s) ds \\ &= u(t_0) + h \int_0^{c_i} u'(t_0 + rh) dr \\ &= u(t_0) + h \int_0^{c_i} \sum_{j=2}^{s-1} l_j(r) K_j dr \quad \text{by (4)} \\ &= u(t_0) + h \sum_{j=2}^{s-1} \int_0^{c_i} l_j(r) K_j dr \end{aligned} \tag{5}$$

Now plugging (5) into (3) and using the definition of $u(t_0)$ as given by (1) we get the following characterization of our K_i's:

$$
\begin{aligned}
K_i &= f(t_0 + c_i h, u(t_0) + h \sum_{j=2}^{s-1} \int_0^{c_i} l_j(r) K_j dr) \\
&= f(t_0 + c_i h, y_0 - h b_1 (u'(t_0) - f(t_0, u(t_0)) + h \sum_{j=2}^{s-1} \int_0^{c_i} l_j(r) K_j dr \quad \text{by (1)} \\
&= f(t_0 + c_i h, y_0 - h b_1 (\sum_{j=2}^{s-1} l_j(0) K_i - f(t_0, u(t_0)) + h \sum_{j=2}^{s-1} \int_0^{c_i} l_j(r) K_j dr \quad \text{by (4)} \\
&= f(t_0 + c_i h, y_0 - h b_1 (\sum_{i=2}^{s-1} l_j(0) K_j - K_1) + h \sum_{j=2}^{s-1} \int_0^{c_i} l_j(r) K_j dr \quad \text{rearranging} \\
&= f(t_0 + c_i h, y_0 + h b_1 K_1 + h \sum_{j=2}^{s-1} K_j (\int_0^{c_i} l_j(r) dr - b_1 l_j(0)) \quad \text{by (3)}
\end{aligned}
$$

Thus as defined, our K_i's are equivalent to those of a RK method with

$$
\begin{cases}
a_{ij} = \int_0^{c_i} l_j(r) dr - b_1 l_j(0) \quad \text{for} \quad j = 2, ..., s-1 \\
a_{i1} = b_1 \\
a_{is} = 0
\end{cases} \tag{6}
$$

We now derive, in a similar way the implied value of our b_i for $i = 2, ...s-1$. Using the FTC once more we derive the following:

$$
\begin{aligned}
u(t_0 + h) &= u(t_0) + \int_{t_0}^{t_0+h} u'(s) ds \\
&= u(t_0) + h \int_0^1 u'(t_0 + rh) dr \\
&= u(t_0) + h \int_0^1 \sum_{j=2}^{s-1} l_j(r) K_j dr \quad \text{by (4)} \\
&= u(t_0) + h \sum_{j=2}^{s-1} \int_0^1 l_j(r) K_j dr
\end{aligned} \tag{7}
$$

Plugging (7) into (2) yields:

$$
\begin{aligned}
y_1 &= u(t_0) + h\sum_{j=2}^{s-1}\int_0^1 l_j(r)K_j dr - hb_s(u'(t_0+h) - f(t_0+h, u(t_0+h)) \\
&= u(t_0) + h\sum_{j=2}^{s-1}\int_0^1 l_j(r)K_j dr - hb_s(u'(t_0+h) - K_s) \quad \textit{By def. of } K_s \\
&= u(t_0) + h\sum_{j=2}^{s-1}\int_0^1 l_j(r)K_j dr - hb_s(\sum_{j=2}^{s-1} l_j(1)K_j - K_s) \quad \textit{By (4)} \\
&= y_0 - hb_1(\sum_{i=2}^{s-1} l_j(0)K_j - K_1) + h\sum_{j=2}^{s-1}\int_0^1 l_j(r)K_j dr - hb_s(\sum_{j=2}^{s-1} l_j(1)K_j - K_s) \quad \textit{By (1) \& (4)} \\
&= y_0 + hb_1K_1 + h\sum_{j=2}^{s-1} K_j\left(\int_0^1 l_j(r)dr - b_s l_j(1) - b_1 l_j(0)\right) + hb_sK_s \qquad (8)
\end{aligned}
$$

Which is exactly a RK method with the following b_i coefficient

$$
bi = \int_0^1 l_i(r)dr - b_s l_i(1) - b_1 l_i(0) \quad \text{for} \quad i = 2, ..., s-1
$$

as needed.

Alternative method

Use the fact that the condition C(s-2) and B(s-2) induces a quadrature method of order s-2. Use them to express the integral ~~the integral~~ of the Legendre polynomials.

Problem 5

We show that the quadrature formula induced by the coefficient of the Lobatto III-B is of order 2s - 3, that is we show:

$$
\int_0^1 q(x)dx = \sum_{i=1}^{s} b_i q(c_i) \quad \forall q \in \mathbb{P}^{2s-3} \qquad (1)
$$

We do so by showing that the LHS, is always equal to the quadrature formula induced by the Lobatto III-A method, which we know is of order 2s-2.

$$
\sum_{i=1}^{s} b_i q(c_i) = b_1 q(c_1) + \sum_{i=2}^{s-1} q(c_i)\left(\int_0^1 l_i(r)dr - b_1 l_i(0) - b_s l_i(1)\right) + b_s q(c_s) \qquad (2)
$$

Where $l_i(x)$ is a Legendre polynomial of order $s-3$ s.t. $l_i(c_j) = \delta_{i,j}$ for $i, j = 2, ..., s-1$. Denoting the coefficient of the Lobatto III-A by $\hat{b_i}$, we express the integral of these polynomial as a sum using the Lobatto III-A quadrature formula. we write:

$$\begin{aligned}\int_0^1 l_i(r)dr &= \sum_{j=1}^{s} l_i(c_j)\hat{b_j} \\ &= l_i(c_1)\hat{b_1} + l_i(c_i)\hat{b_i} + l_i(c_s)\hat{b_s} \quad l_i(c_j) = \delta ij \\ &= l_i(0)b_1 + \hat{b_i} + l_i(1)b_s \quad \textit{By def} \quad \hat{b_1} = b_1 \;\&\; \hat{b_s} = b_s \end{aligned} \tag{3}$$

Plugging in this result into (2) we get:

$$\begin{aligned}\sum_{i=1}^{s} b_i q(c_i) &= b_1 q(c_1) + \sum_{i=2}^{s-1} q(c_i)\hat{b_i} + b_s q(c_s) \\ &= \int_0^1 q(x)dx \quad \textit{By order of the Lobatto III-A}\end{aligned} \tag{4}$$

as needed